ACT 3230 Actuarial Models II

Exam 2 – Chapter 8

March 11, 2009
4:00 p.m. – 5:15 p.m.

Instructor: Sheldon Liu, FSA, FCIA

1. Rorschach purchases a special 4-year term insurance policy from the New-Insurance-To-Enhance-Our-Wealth Life Insurance Company (NITE OWL). The benefit under this insurance, payable at the end of the year of death, varies with the year of death. You are given the following information on the benefit payment and the probability function for $K(x)$, Rorschach's curtate-future-lifetime:

K	Benefit Payment	Probability of death: ${}_{k|}q_x$
0	400	0.1
1	300	0.2
2	200	0.3
3	100	0.4

Rorschach makes 4 level premium payments of 61.68 (calculated based on the equivalence principle) at the beginning of each year he is alive, starting at policy issue. The effective interest rate is 6% per year.

Assuming Rorschach is alive at the end of year 2, calculate the benefit reserve on this policy at that time, just *before* he makes his third premium payment.

<u>Solution</u>:

Note that the unconditional probabilities of death (${}_{k|}q_x$) are provided, and we need q_{x+k}.
Re: ${}_{k|}q_x = {}_kp_x\, q_{x+k}$

k	${}_{k|}q_x$	$q_{x+k} = {}_{k|}q_x \,/\, {}_kp_x$	$p_{x+k} = 1 - q_{x+k}$	${}_kp_x = \Pi\, p_{x+k}$
0	0.1	0.1	0.9	0.9
1	0.2	2/9	7/9	0.7
2	0.3	3/7	4/7	0.4
3	0.4	1	0	0

Since premiums are calculated based on the equivalence principle, ${}_0V = 0$.
Re: $({}_hV + \pi_h)(1+i) = b_{h+1} \cdot q_{x+h} + {}_{h+1}V \cdot p_{x+h}$

So, $$\frac{({}_hV + \pi_h)(1+i) - b_{h+1} \cdot q_{x+h}}{p_{x+h}} = {}_{h+1}V$$

$${}_1V = \frac{(0 + 61.68)(1.06) - 400(0.1)}{(1-0.1)} = 28.2009$$

$${}_2V = \frac{(28.2009 + 61.68)(1.06) - 300(2/9)}{(1-2/9)} = \mathbf{36.78}$$

Alternatively, use the prospective method:

APV of FB $= b_3 v^1 q_{x+2} + b_4 v^2 q_{x+3}$
$= (200)(1/1.06)(3/7) + (100)(1.06)^{-2}(4/7)(1)$
$= 131.72$

APV of FP $= \pi_2 + \pi_3 v p_{x+2}$
$= 61.68 + 61.68(1/1.06)(4/7)$
$= 94.93$

$_2V$ = APV of FB – APV of FP = 131.72 – 94.93 = **36.79**

2. A 15-year term insurance policy has level benefit premiums payable during the lifetime of the policy. The policy is issued at age 65 and has a death benefit of $1,000. You are given:
   i) $i = 0.08$
   ii) $q_{78} = 0.0645$
   iii) $\text{Var}[{}_{13}L|K \geq 13] = 92{,}503.12$
   iv) $\text{Var}[{}_{14}L|K \geq 14] = 54{,}334.71$

   Calculate $1000 \cdot {}_{14}V^{1}_{65:\overline{15|}}$.

   Solution:
   $\text{Var}[{}_{13}L|K \geq 13] = v^2 (1000 - 1000\, {}_{14}V^{1}_{65:\overline{15|}})^2 p_{78}\, q_{78} + v^2 p_{78} \text{Var}[{}_{14}L|K \geq 14]$

   $92{,}503.12 = (1.08)^{-2} (1000 - 1000\, {}_{14}V^{1}_{65:\overline{15|}})^2 (1 - .0645)(.0645) + (1.08)^{-2} (1 - .0645)(54{,}334.71)$

   $\rightarrow 1000\, {}_{14}V^{1}_{65:\overline{15|}} = 1000 - \sqrt{\dfrac{(1.08)^2}{(.0645)(.9355)} \times \left[92{,}503.12 - \dfrac{(54{,}334.71)(.9355)}{(1.08)^2}\right]} = \mathbf{27.51}$

3. For a special 3-year deferred life annuity-due on (x), you are given:
   i) The first annual payment is 500.
   ii) Subsequent annual payments increase 5% per year.
   iii) There are no death benefits.
   iv) Level annual benefit premiums are payable for 3 years.
   v) $i = 0.05$
   vi)

n	e_{x+n}	p_{x+n}
0	30.3	Not given
1	29.6	Not given
2	29.2	0.97

   Calculate the benefit reserve at the end of year 2.

   Hint: $e_x = p_x (1 + e_{x+1})$

Solution:
Let P = annual premium

At time 0, APV of FP = APV of FB:

$$P\,\ddot{a}_{x:\overline{3}|} = 500v^3{}_3p_x \sum_{k=0}^{\infty}(1.05)^k v^k\,{}_kp_{x+3} = 500v^3{}_3p_x \sum_{k=0}^{\infty}{}_kp_{x+3} = 500v^3{}_3p_x\,(1 + e_{x+3})$$

$e_x = p_x\,(1 + e_{x+1}) \rightarrow p_x = 0.9902$
$e_{x+1} = p_{x+1}\,(1 + e_{x+2}) \rightarrow p_{x+1} = 0.9801$
$p_{x+2} = 0.97$
${}_3p_x = p_x \cdot p_{x+1} \cdot p_{x+2} = 0.9414$
$e_{x+2} = p_{x+2}\,(1 + e_{x+3}) \rightarrow e_{x+3} = 29.1$
$\ddot{a}_{x:\overline{3}|} = 1 + vp_x + v^2{}_2p_x = 2.823$

$P = [500(1.05)^{-3}(.9414)(1 + 29.1)] / 2.823 = 4{,}335.39$

$${}_2V = P\,\ddot{s}_{x:\overline{2}|} = 4{,}335.39 \cdot \frac{1 + vp_x}{v^2\,{}_2p_x} = \mathbf{9{,}569}$$

The above uses the retrospective formula, recognizing that COI = 0.

Using $({}_hV + \pi_h)(1+i) = b_{h+1} \cdot q_{x+h} + {}_{h+1}V \cdot p_{x+h}$ would get to the same result:

$({}_0V + \pi)(1+i) = b_1 \cdot q_x + {}_1V \cdot p_x$ // $b_1 = 0,\ {}_0V = 0$

$\pi(1+i) = {}_1V \cdot p_x$

$${}_1V = \frac{\pi(1+i)}{p_x} = \frac{\pi}{vp_x}$$

$({}_1V + \pi)(1+i) = b_2 \cdot q_{x+1} + {}_2V \cdot p_{x+1}$

$(\frac{\pi}{vp_x} + \pi)(1+i) = {}_2V \cdot p_{x+1}$

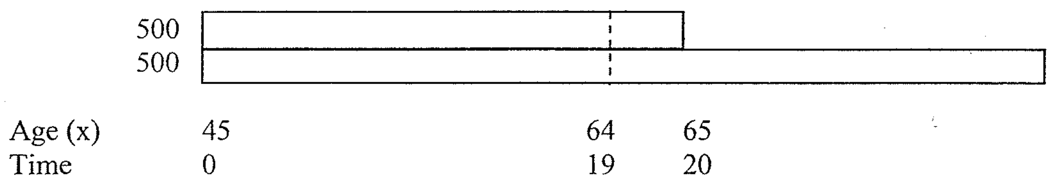
$${}_2V = \frac{(\frac{\pi}{vp_x} + \pi)}{vp_{x+1}} = \pi\,\ddot{s}_{x:\overline{2}|} = 4{,}335.39 \cdot \frac{1 + vp_x}{v^2\,{}_2p_x} = \mathbf{9{,}569}$$

4. For a special fully discrete life insurance on (45), you are given:
   i) $i = 6\%$
   ii) Mortality follows the Illustrative Life Table
   iii) The death benefit is 1000 until age 65, and 500 thereafter
   iv) Benefit premiums of 12.51 are payable at the beginning of each year for 20 years.

   Calculate ${}_{19}V$, the benefit reserve at time $t = 19$ (the instant before the premium payment is made).

Solution:
${}_{19}V$ = APV of future benefit – APV of future premium

Future benefit diagram:



APV of future benefit $= 500vq_{64} + 500A_{64}$
$= 500(1/1.06)(.01952) + 500(.42522) = 221.82$

APV of future premium = 12.51 (only 1 more premium payment left; due immediately)

${}_{19}V = 221.82 - 12.51 =$ **209.31**

5. A fully discrete whole life insurance with face amount 100,000 and level premiums is issued at age 80. Mortality follows the Illustrative Life Table at 6% interest.

Let ${}_{10.5}V_{80}^{UDD}$ = the benefit reserve at time 10.5 assuming UDD over the age interval,
${}_{10.5}V_{80}^{NO}$ = the benefit reserve at time 10.5 assuming $i = 0$ and $q_{90} = 0$

Calculate ${}_{10.5}V_{80}^{UDD} - {}_{10.5}V_{80}^{NO}$ (the difference between the two approximations).

Solution:
From the Illustrative Life Table,
$A_{80} = 0.66575,\ \ddot{a}_{80} = 5.9050,\ \ddot{a}_{90} = 3.6488,\ \ddot{a}_{91} = 3.4611,\ q_{90} = .18877$

$${}_{10}V_{80} = 1 - \frac{\ddot{a}_{90}}{\ddot{a}_{80}} = 1 - \frac{3.6488}{5.9050} = .3820830$$

$${}_{11}V_{80} = 1 - \frac{\ddot{a}_{91}}{\ddot{a}_{80}} = 1 - \frac{3.4611}{5.9050} = .4138696$$

Note: Each of these is multiplied by 100,000 below.

Prem = $100{,}000\, P_{80} = 100{,}000\,(A_{80}/\ddot{a}_{80}) = 100{,}000\,(.66575 / 5.9050) = 11{,}274.34$

Under UDD,
${}_{0.5}p_{90} = 1 - {}_{0.5}q_{90} = 1 - 0.5(q_{90}) = 1 - 0.5(.18877) = 0.905615$

$v^{0.5}\ {}_{0.5}p_{90}\ {}_{10.5}V_{80}^{UDD} = ({}_{10}V_{80} + \text{Prem})(1 - 0.5) + ({}_{11}V_{80}\, v\, p_{90})(0.5)$

$${}_{10.5}V_{80}^{UDD} = \frac{[(38{,}208.30 + 11{,}274.34)(0.5) + (41{,}386.96)(1.06)^{-1}(1 - .18877)(0.5)]}{[0.905615 / (1.06)^{0.5}]}$$

$= 46{,}132.07$

Under the approximation where $i = 0$ and $q_{90} = 0$,

$$\begin{aligned} {}_{10.5}V_{80}^{NO} &= (0.5)({}_{10}V_{80} + \text{Prem}) + (0.5)({}_{11}V_{80}) \\ &= (0.5)[38{,}208.30 + 11{,}274.34 + 41{,}386.96] \\ &= 45{,}434.80 \end{aligned}$$

$${}_{10.5}V_{80}^{UDD} - {}_{10.5}V_{80}^{NO} = 46{,}132.07 - 45{,}434.80 = \mathbf{697.27}$$

6. For a special fully discrete whole life insurance on (40), you are given:
   i) The level premium for this insurance is equal to P_{20}.
   ii) ${}_kV = {}_kV_{20},\ k = 0, 1, \ldots, 19$
   iii) ${}_{11}V = {}_{11}V_{20} = 0.08154$
   iv) $q_{40+k} = q_{20+k} + 0.01,\ k = 0, 1, \ldots, 19$
   v) $q_{30} = 0.008427$

   Calculate b_{11}, the death benefit in year 11.

Solution:

Re: $({}_hV + \pi_h)(1+i) = b_{h+1} \cdot q_{x+h} + {}_{h+1}V \cdot p_{x+h}$

For the regular fully discrete whole life insurance on (20),

$$\begin{aligned} ({}_{10}V_{20} + P_{20})(1+i) &= (1)(0.008427) + (0.08154)(1 - 0.008427) \\ ({}_{10}V_{20} + P_{20})(1+i) &= 0.089280 \end{aligned}$$

For the special fully discrete whole life insurance on (40),

$$\begin{aligned} ({}_{10}V + P_{20})(1+i) &= (b_{11})(0.018427) + (0.08154)(1 - 0.018427) \\ 0.089280 &= (b_{11})(0.018427) + 0.080037 \\ b_{11} &= \mathbf{0.501576} \end{aligned}$$

7. Assuming a uniform distribution of deaths in each year of age, you are given:
   i) ${}_hV^{(2)} = 0.3$
   ii) ${}_{h+1}V^{(2)} = 0.5$
   iii) $\pi_h^{(2)} = 0.4$
   iv) $v = 0.95$
   v) $p_{x+h} = 0.90$

   Determine the benefit reserve at the end of the first month of the year, ${}_{h+1/12}V^{(2)}$

Solution:

Re: Under UDD, ${}_sq_x = s \cdot q_x$

$${}_{1/12}p_{x+h} = 1 - {}_{1/12}q_{x+h} = 1 - (\tfrac{1}{12}) \cdot q_{x+h} = 1 - (1/12)(1 - 0.90) = 0.99166$$

$${}_{0.5}p_{x+h} = 1 - {}_{0.5}q_{x+h} = 1 - (0.5) \cdot q_{x+h} = 1 - (0.5)(1 - 0.90) = 0.95$$

$$v^s\,{}_sp_{x+h}\;{}_{h+s}V^{(2)} = ({}_hV^{(2)} + \frac{\pi_h^{(2)}}{2})(1-s) + [{}_{h+1}V^{(2)}\,v\,p_{x+h} - \frac{\pi_h^{(2)}}{2}(v^{0.5})\,{}_{0.5}p_{x+h}](s)$$

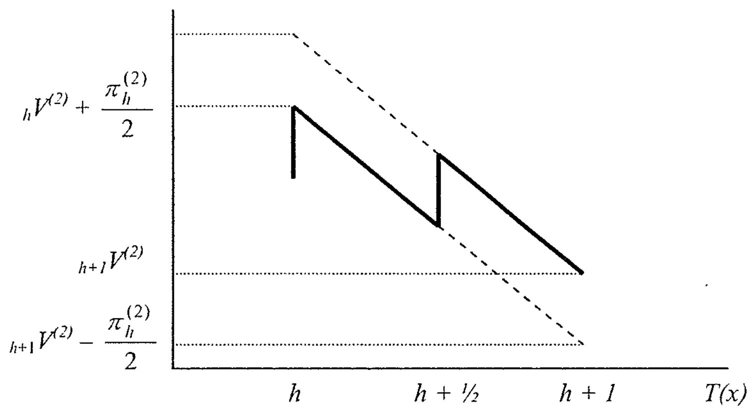
$$(.95)^{1/12}(0.99166)\,{}_{h+1/12}V^{(2)} = (0.3 + \frac{0.4}{2})(1 - 1/12) + [0.5(.95)(.90) - \frac{0.4}{2}(.95)^{0.5}(0.95)](1/12)$$

$$(0.987436892)\;{}_{h+1/12}V^{(2)} = (0.5)(11/12) + (0.242310907)(1/12)$$

$${}_{h+1/12}V^{(2)} = 0.478525909 / 0.987436892 = \mathbf{0.484614169}$$

Generic graph:
(no i or q_{x+h})

Think about which 2 points you are interpolating between (depending on which half of the year you are in), and then factor in i and/or q_{x+h} as necessary.



8. A special 20-year fully discrete endowment insurance policy is paid for annually by 10 level benefit premiums starting on the policy issue date. The death benefit is the terminal benefit reserve for that year, and the endowment is \$1000. If i = 0.04, what is ${}_5V$?

Solution:

Re: $${}_kV = \sum_{h=0}^{k-1}\left[\pi_h - (b_{h+1} - {}_{h+1}V)vq_{x+h}\right](1+i)^{k-h}$$

And $\pi_h = (b_{h+1} - {}_{h+1}V)vq_{x+h} + (v\,{}_{h+1}V - {}_hV)$

Since $b_{h+1} = {}_{h+1}V$,

$\pi = v\,{}_{h+1}V - {}_hV$

Multiply by v^h: $\pi\, v^h = v^{h+1}\,{}_{h+1}V - v^h\,{}_hV$

On both sides, do $\sum_{h=0}^{k-1}$: $\pi\,\ddot{a}_{\overline{k}|} = v^k\,{}_kV - v^0\,{}_0V$

So, ${}_kV = \pi\,\ddot{s}_{\overline{k}|}$

$$_{20}V = 1000 = \pi \, \ddot{s}_{\overline{20}|.04}$$
$$\pi = 54.107$$
$$_{5}V = \pi \, \ddot{s}_{\overline{5}|.04} = (54.107)(5.633) = \mathbf{304.78}$$

9. For a fully discrete whole life insurance with level annual premiums and level death benefit on the life of (x), you are given:
   i) $i = 0.05$
   ii) $q_{x+h-1} = 0.004$
   iii) The initial benefit reserve for policy year h is 200
   iv) The net amount at risk for policy year h is 1295
   v) $\ddot{a}_x = 16.2$

   Calculate the terminal reserve for policy year $h - 1$.

   Solution:

   $$({}_{h-1}V + \pi_{h-1})(1 + i) = b_h \, q_{x+h-1} + {}_hV \, p_{x+h-1}$$
   $$= (b_h - {}_hV) \, q_{x+h-1} + {}_hV$$
   $$\rightarrow {}_hV = (200)(1.05) - 1295(0.004) = 204.82$$

   $$\text{NAAR} = b_h - {}_hV = 1295 \rightarrow b_h = 1500$$

   At issue, $1500A_x = \pi \ddot{a}_x$

   $$\rightarrow \pi = 1500P_x = 1500\left(\frac{1}{\ddot{a}_x} - d\right) = 21$$

   $${}_{h-1}V + \pi_{h-1} = 200 \rightarrow {}_{h-1}V = 200 - 21 = \mathbf{179}$$

10. On January 1, 2007, Insurance Izz Us (IIU) issued a fully discrete whole life insurance policy to (x). Investing a majority of its money in mortgage-backed securities, IIU was in trouble 2 years later as the world economy collapsed. IIU decided to close, so deleted all policyholder information from its systems.

    A day later, the government promised $30 billion to IIU as bailout relief. IIU quickly decided to open its doors again. Thankfully, a summer student found some paper copies containing information on (x)'s policy, to be used to set up the proper reserve at March 31, 2009:
    i) Benefit = 1000
    ii) This year's annual premium = 50
    iii) Initial benefit reserve on January 1, 2009 = 500

    Assuming i = 6%, $q_{x+2} = 0.1$, and a uniform distribution of deaths over each year of age, find ${}_{2.25}V$.

Solution:

Re: Under UDD, ${}_sq_{x+h} = (s)(q_{x+h})$

$$({}_hV + \pi_h)(1+i)^s = b_{h+1} \cdot v^{1-s} \cdot {}_sq_{x+h} + {}_{h+s}V \cdot {}_sp_{x+h}$$

$(500)(1.06)^{0.25} = (1000)(1.06)^{-0.75}(.25)(.1) + [1 - .25(.1)]\ ({}_{2.25}V)$

$507.3369231 = 23.93098694 + (.975)\ ({}_{2.25}V)$

${}_{2.25}V = \mathbf{495.8009602}$

Note: This year's annual premium is not necessary, as it is already included in the initial benefit reserve.

Alternative:

$({}_hV + \pi_h)(1+i) = b_{h+1} \cdot q_{x+h} + {}_{h+1}V \cdot p_{x+h}$

$(500)(1.06) = (1000)(.1) + (.9)({}_{h+1}V)$

${}_{h+1}V = 477.7777777$

Re: Under UDD, ${}_sq_{x+t} = \dfrac{s \cdot q_x}{1 - t \cdot q_x}$

$${}_{h+s}V = b_{h+1} \cdot v^{1-s} \cdot {}_{1-s}q_{x+h+s} + {}_{1-s}p_{x+h+s} \cdot v^{1-s}\ {}_{h+1}V$$

$$\begin{aligned} {}_{2.25}V &= (1000(1.06)^{-0.75}\left[\frac{.75(.1)}{1-.25(.1)}\right] + \left[1 - \frac{.75(.1)}{1-.25(.1)}\right](1.06)^{-0.75}(477.7777777) \\ &= 73.63380596 + 422.1671542 \\ &= \mathbf{495.8009602} \end{aligned}$$

Alternative:

As above, solve for:

${}_{h+1}V = 477.7777777$

Under UDD,

$v^s\ {}_sp_{x+h}\ {}_{h+s}V = ({}_hV + \pi_h)(1-s) + ({}_{h+1}V\ \ v\,p_{x+h})(s)$

$(1.06)^{-.25}[1 - 0.25(.1)]\ {}_{h+s}V = (500)(1 - 0.25) + (477.777)(1/1.06)(1 - 0.1)(0.25)$

${}_{h+s}V = (375 + 101.4150943) / 0.960899903 = \mathbf{495.8009602}$